A forward-backward diffusion model for geomorphological generalization and image processing

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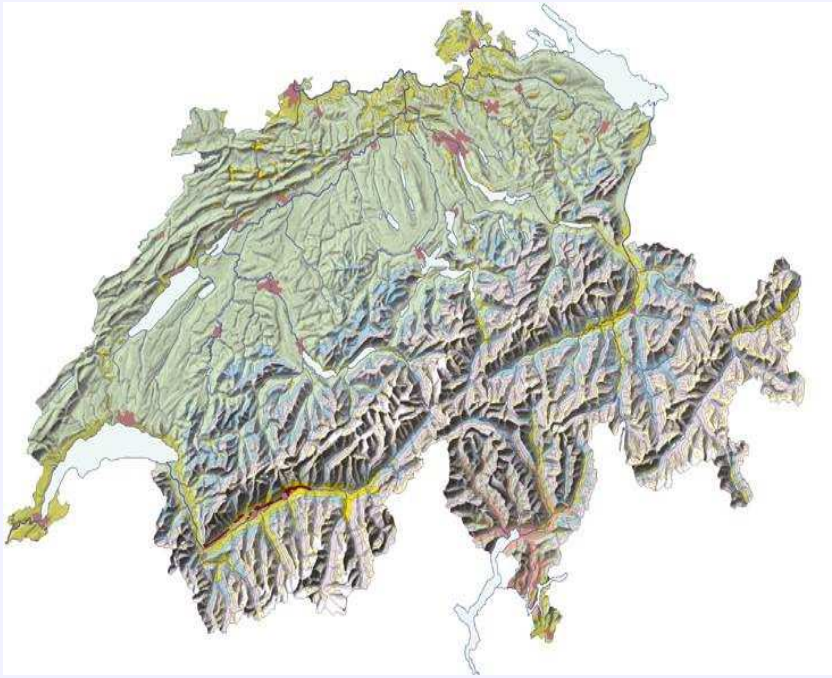
High mountain regions present hotspots of biodiversity. They exploit the third dimension in a small-sized area creating different climatic conditions over short distances and therefore a high biodiversity.

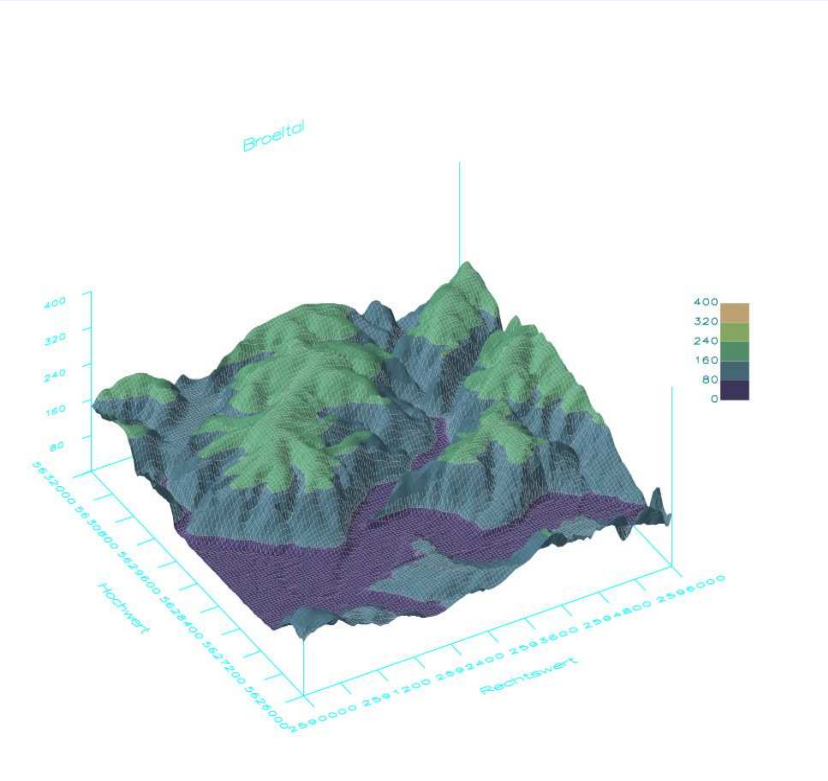
- Relation between phyto (plant)-diversity at different scales and landform
- To a certain extent geomorphometric parameters can describe plant diversity

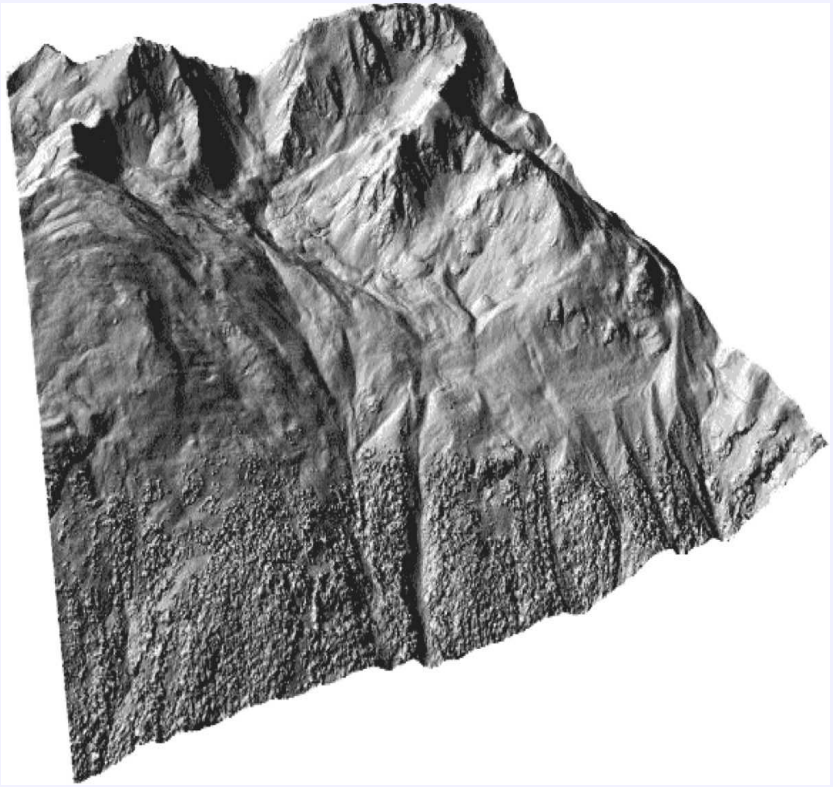


Kleinod, K., Wissen, M., Bock, M. (2005)

Digital Elevation Models

Digital Elevation Models (DEM) are the most basic and interesting geographical data type. A Digital Elevation Model is an ASCII or binary file that contains only spatial elevation data in a regular gridded pattern in raster format.





Outline

1. Diffusion models for generalization
2. Mathematical tool: Young-measure solution
3. Finite-Element-Approximation and numerical simulation

Form Generalization

Aims

- to simplify in a reasonable way
- reduce complexity (number of variables can easily be in the billions)
- to identify the characteristic shapes
- to preserve the edge information

Main Tasks

- edge enhancement
- selective smoothing
- noise removal

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Diffusion model for Generalization

Model based on a forward-backward equation. Proposed by Perona und Malik ('90) in image processing.

$$\begin{aligned} t &= \text{scale parameter, regular parameter} \\ u(x,t) &= \text{Elevation at the space point } x \text{ in dependence of } t \end{aligned}$$

The DEM gets generalized, which means selective smoothing

$$u_t(x,t) - \operatorname{div}(a(|\nabla u(x,t)|)\nabla u(x,t)) = 0, \quad x \in \mathbb{R}^2$$

Diffusion coefficient is controlled by the modulus of the gradient

$$a(|\nabla u|) :\approx \begin{cases} 0 & \text{for} \quad |\nabla u| \text{ large} \\ 1 & \text{for} \quad |\nabla u| \text{ small} \end{cases}$$

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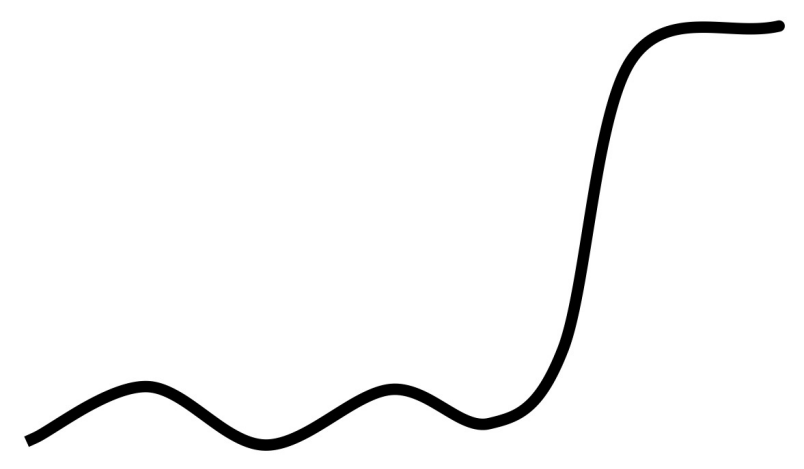
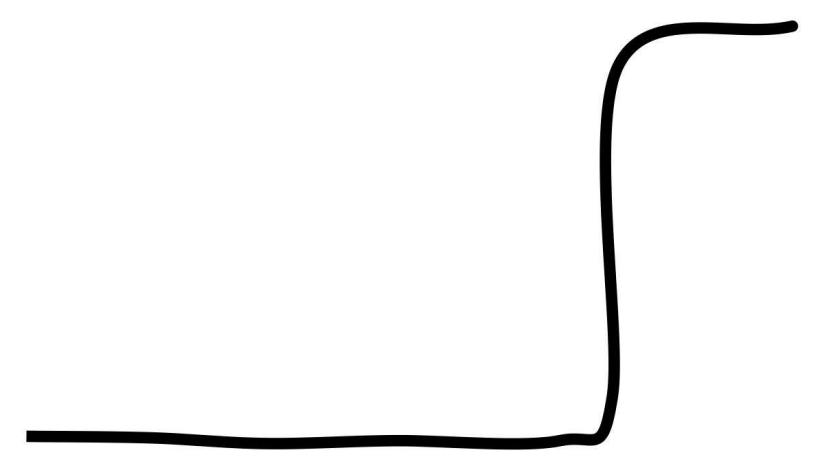
Principal Idea

$u_t = (a\,u_x)_x$

- $a = 1$: $\quad u_t = u_{xx}$
- $a = 0$: $\quad u_t = 0$





Diffusion Equation

$u(x,t)$ solves

$$\frac{\partial u}{\partial t} = \operatorname{div}\left[a(|\nabla u|)\nabla u\right] \quad \text{in } \Omega \times (0,T]$$

$$a(|\nabla u|)\,\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T]$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega$$

where:

- Bounded area $\Omega \subset \mathbb{R}^2$
- Function $u(\cdot,t) : \Omega \to \mathbb{R}_+$ denotes the shape and $t \in \mathbb{R}_+$ the regularization parameter
- The DEM u_0 as initial value

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Examples:

- Heat equation (Gaussian filter)

$$\frac{\partial u}{\partial t} = \Delta u$$

- Regularized Perona-Malik equation

$$\frac{\partial u}{\partial t} = \mathrm{div}\,[(1 + |\nabla G_\sigma * u|^2)^{-1} \nabla u]$$

[Catte, Lions, Morel, Coll]
[Aubert, Kornprobst]

TV-Flow

Total Variation model:

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{1}{|\nabla u|}\nabla u\right) .$$

- Monotonicity:

$$\big(a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v\big) \cdot \big(\nabla u - \nabla v\big) \geq 0$$

  where $a(|\nabla u|) = \frac{1}{|\nabla u|}$.

- Monotonicity $\Rightarrow$ Convexity $\Rightarrow$ Existence of solution
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Perona-Malik

Perona-Malik model (1990)

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\lambda}{\lambda + |\nabla u|^2}\nabla u\right)$$

- *Controlled diffusivity*

$$a(|\nabla u|) = \frac{\lambda}{\lambda + |\nabla u|^2}$$

  with control parameter $\lambda > 0$
- *Degenerate equation*

$$a(|\nabla u|)\nabla u \to 0 \quad \text{for} \quad |\nabla u| \to \infty$$

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forward diffusion, well posed

- If $|\nabla u|, |\nabla v|$ are large then

$$\big(a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v\big) \cdot \big(\nabla u - \nabla v\big) \leq 0$$

backward diffusion, ill posed

Examples

$$\frac{\partial u}{\partial t} - \Delta u = 0 \qquad \text{(forward diffusion)}$$

$$\frac{\partial u}{\partial t} + \Delta u = 0 \qquad \text{(backward diffusion)}$$

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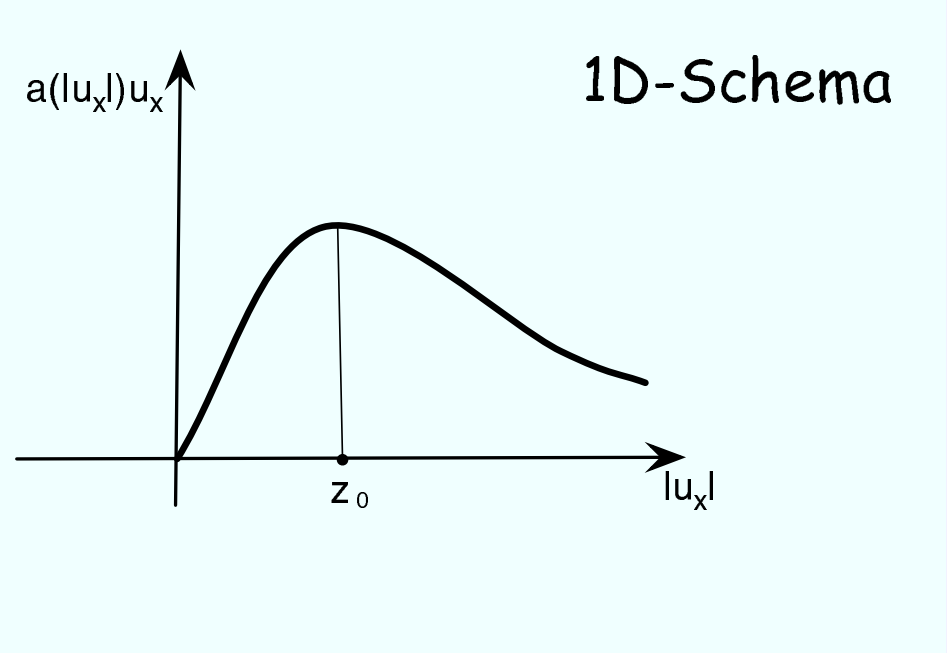

Model is characterized by

- $a(|s|) \geq 0$
- λ defines a critical value z_0:

$$\partial_s(a(|s|)\,s) = \begin{cases} > 0 & \text{for } |s| < z_0 \\ < 0 & \text{for } |s| > z_0 \end{cases}$$

- $a(|s|)s \to 0$ and $\partial_s(a(|s|)\,s) \to 0$ for $s \to \pm\infty$

The Perona-Malik equation is

- nonlinear
- forward + backward
- degenerated

Paradoxon:

- Non-existence results of solutions
  [Kawohl, Kutev '98]: $\nexists$ C^1–solutions
  [Kichenassamy '97]: $\nexists$ weak solutions
- Numerical schemes provide excellent results

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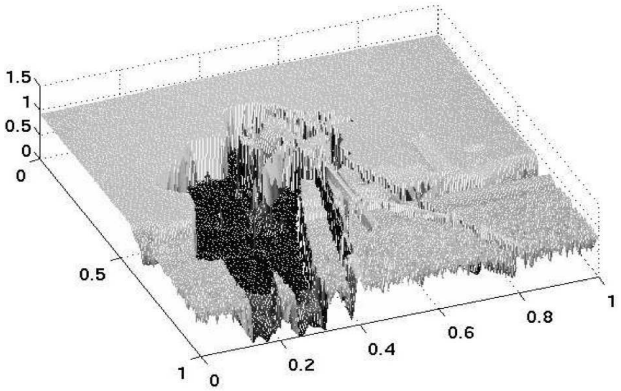
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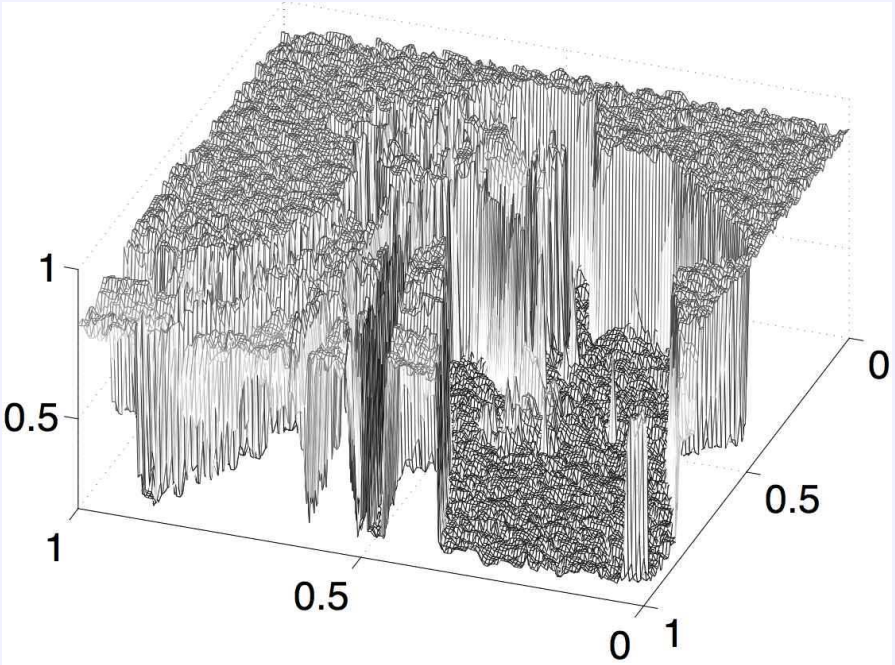
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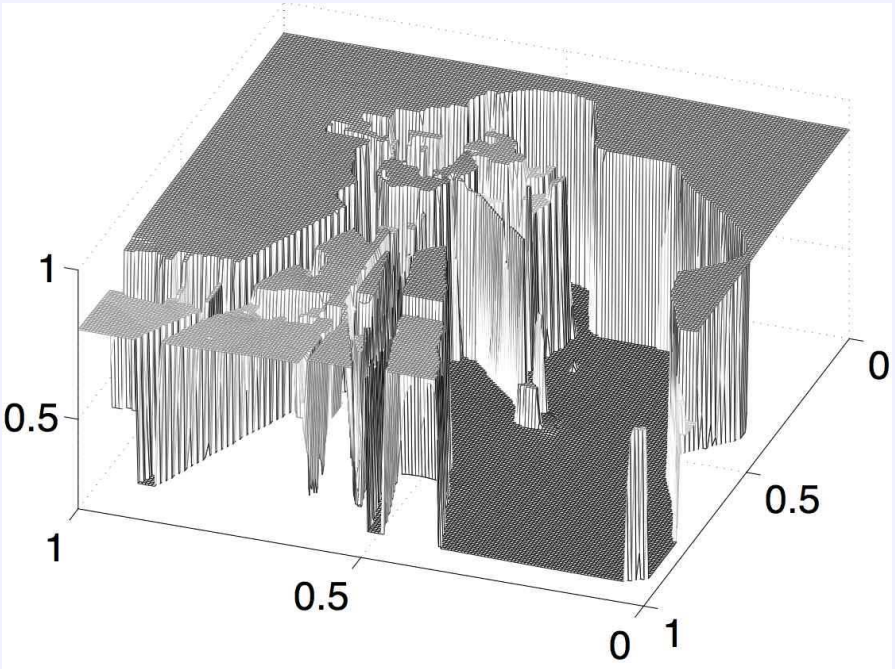
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Anisotropic Diffusion model

V. (2006)

$$u_t - \operatorname{div}\left(\frac{\lambda|\nabla u|^{2\delta}}{\lambda + |\nabla u|^{2(\delta+1)}}\nabla u\right) - \mu \underbrace{\operatorname{div}\left(\frac{|\nabla u|^{3\delta+1}}{\lambda + |\nabla u|^{2(\delta+1)}}\nabla u\right)}_{\text{with } p=1+\delta-\text{Structure}} = 0$$

for $0 < \mu << 1$, $\delta \geq 0$ and $\lambda > 0$

Case $\delta = 0$:

Ebmeyer and V. (2005):

$$u_t - \operatorname{div}\left(\frac{\lambda + \mu|\nabla u|}{\lambda + |\nabla u|^2}\,\nabla u\right) = 0 \qquad (\lambda, \mu > 0)\,.$$

The function

$$s \to \frac{\lambda + \mu s}{\lambda + s^2}\, s$$

- has its maximum at $s_0 = \mu + \sqrt{\lambda + \mu^2}$,
- is monotonously increasing in $(0, s_0)$
- is monotonously decreasing in (s_0, ∞)

$\Rightarrow$ forward and backward diffusion

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Limit cases

- $\lambda = 0$: Total Variation model

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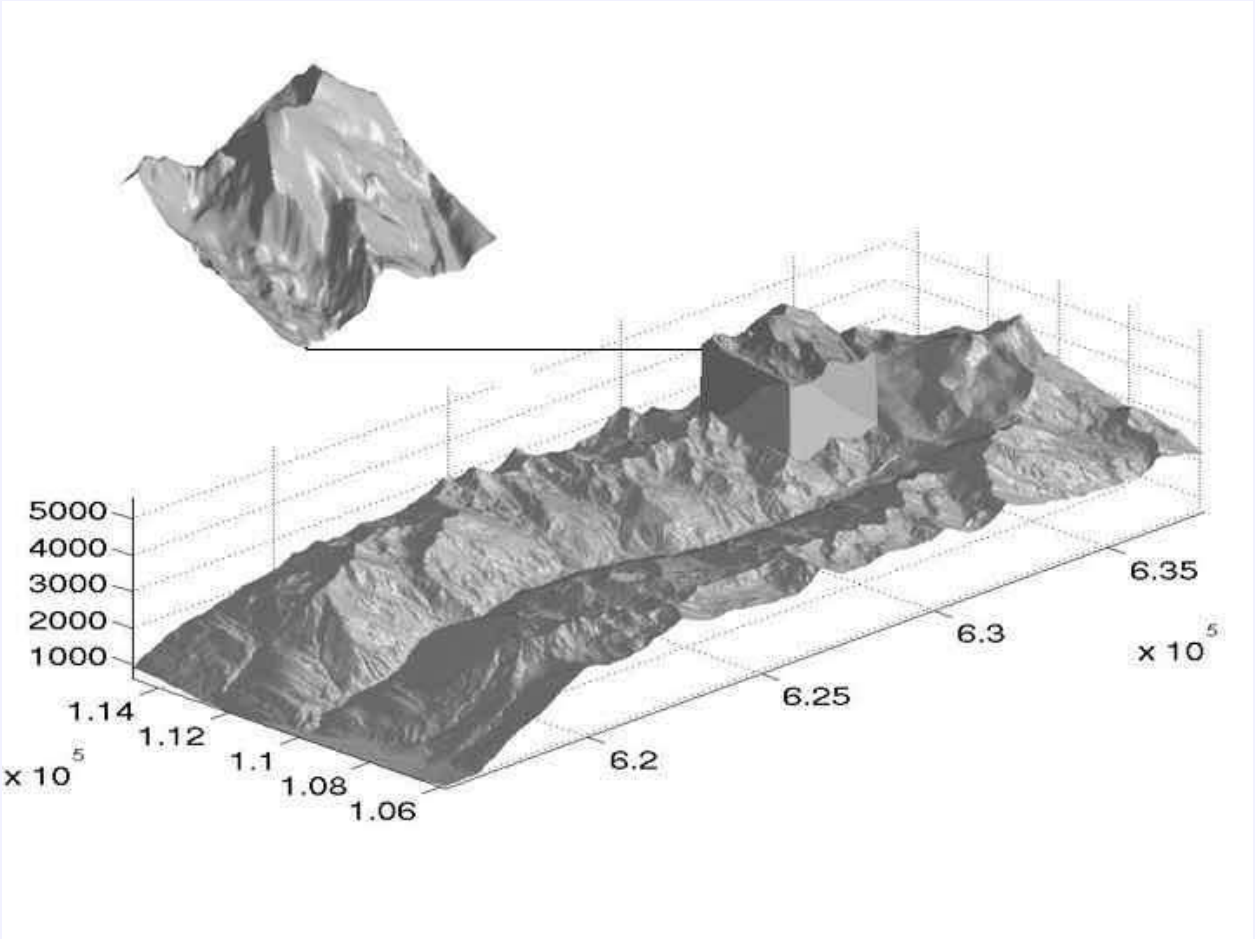

'Anisotropic' diffusion

$$u_t - \operatorname{div}\left(a(|\nabla u|)\,\nabla u\right) = 0$$

Let

$\eta = -\nabla u/|\nabla u|$ be the direction of the steepest descent

τ be a direction tangent to ∇u.

Then

$$u_t - a(|\nabla u|)\,\Delta u - a'(|\nabla u|)\,|\nabla u|\,\partial_{\eta\eta} u = 0.$$

Therefore

$$u_t - a(|\nabla u|)\,\partial_{\tau\tau} u - b(|\nabla u|)\,\partial_{\eta\eta} u = 0$$

where $a > 0$ and b switches sign.

Existence of Young measure solution

Nonconvexity: Lack of classical solutions

Bolza problem

$$\inf_u I(u) := \int_{-1}^{1} \left(u'(x)^2 - 1\right)^2 + u(x)^2 \, dx$$

where

$$u : (0,1) \to \mathbb{R} \ \in W^{1,4}(0,1)$$
$$u(0) = u(1) = 0$$

- Saw-tooth functions $u_j(x) := \frac{1}{j} u(jx)$, where

  $$u(y) := |y| \quad y \in [-0.5, 0.5]$$

  extended periodically on $\mathbb{R}$.

- $u := \lim_{j\to\infty} u_j$ should be a solution
- $lim_{j\to\infty} I(u_j) = 0$
- But $I(v) = 0$ is impossible for a single function!

Weak Formulation

$$\int_\Omega \frac{\partial u}{\partial t}\phi\, dx + \int_\Omega \mathbf{q}(\nabla u)\nabla\phi\, dx = 0 \quad \forall \phi \in V$$

where

$$\mathbf{q}(s) := a(|s|)s$$

- If **q** has no montone structure => There exists no weak solution in general
- New definition of a solution: measure value gradient
- Tartar ('79), DiPerna ('83), Ball ('89), Kinderleher & Pedegral ('92), ...

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Weak Formulation

Solution $(u, \nu_{x,t})$ consists of a regular part u and a measure value part $\nu_{x,t}$ such that:

$$\int_\Omega \frac{\partial u}{\partial t}\phi\, dx + \int_\Omega \int_{\mathbb{R}^2} \mathbf{q}(\vec{\xi})\nabla\phi\, d\nu_{x,t}(\vec{\xi})\, dx = 0 \quad \forall \phi \in V$$

where

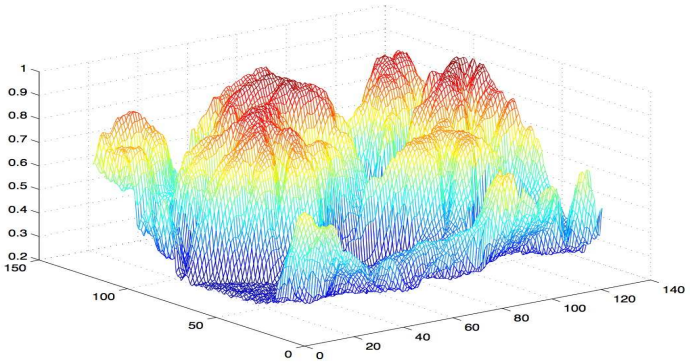
$$\nabla u(x,t) = \int_{\mathbb{R}^2} \vec{\xi}\, d\nu_{x,t}(\vec{\xi}) \quad \text{for almost all } x \in \Omega$$

for almost all t.

Numerical Simulation

Phenomena

- Invariance property
- Convexification effect
- Rounding effect
- Extinction in finite time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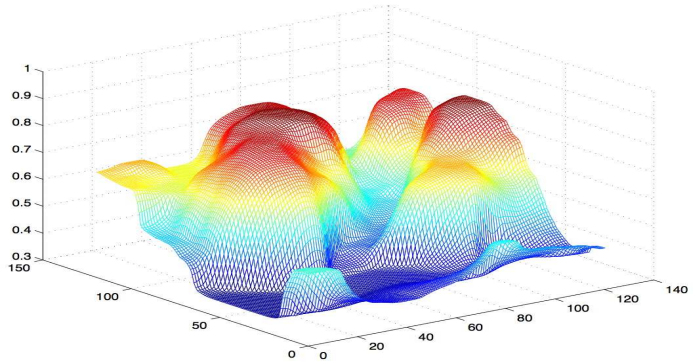
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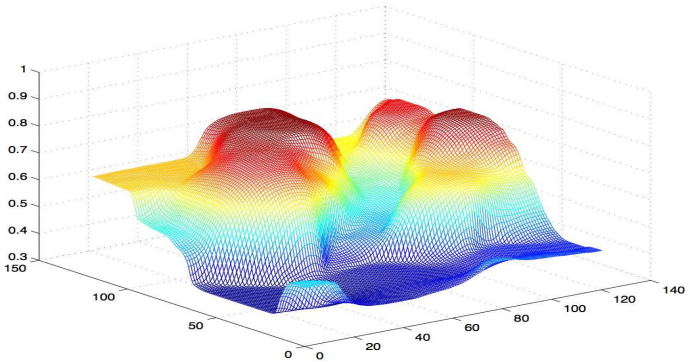
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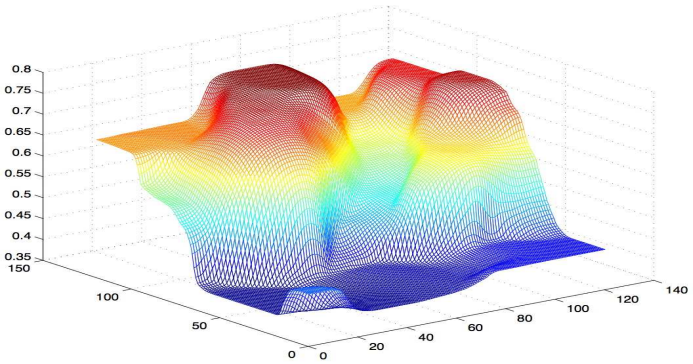
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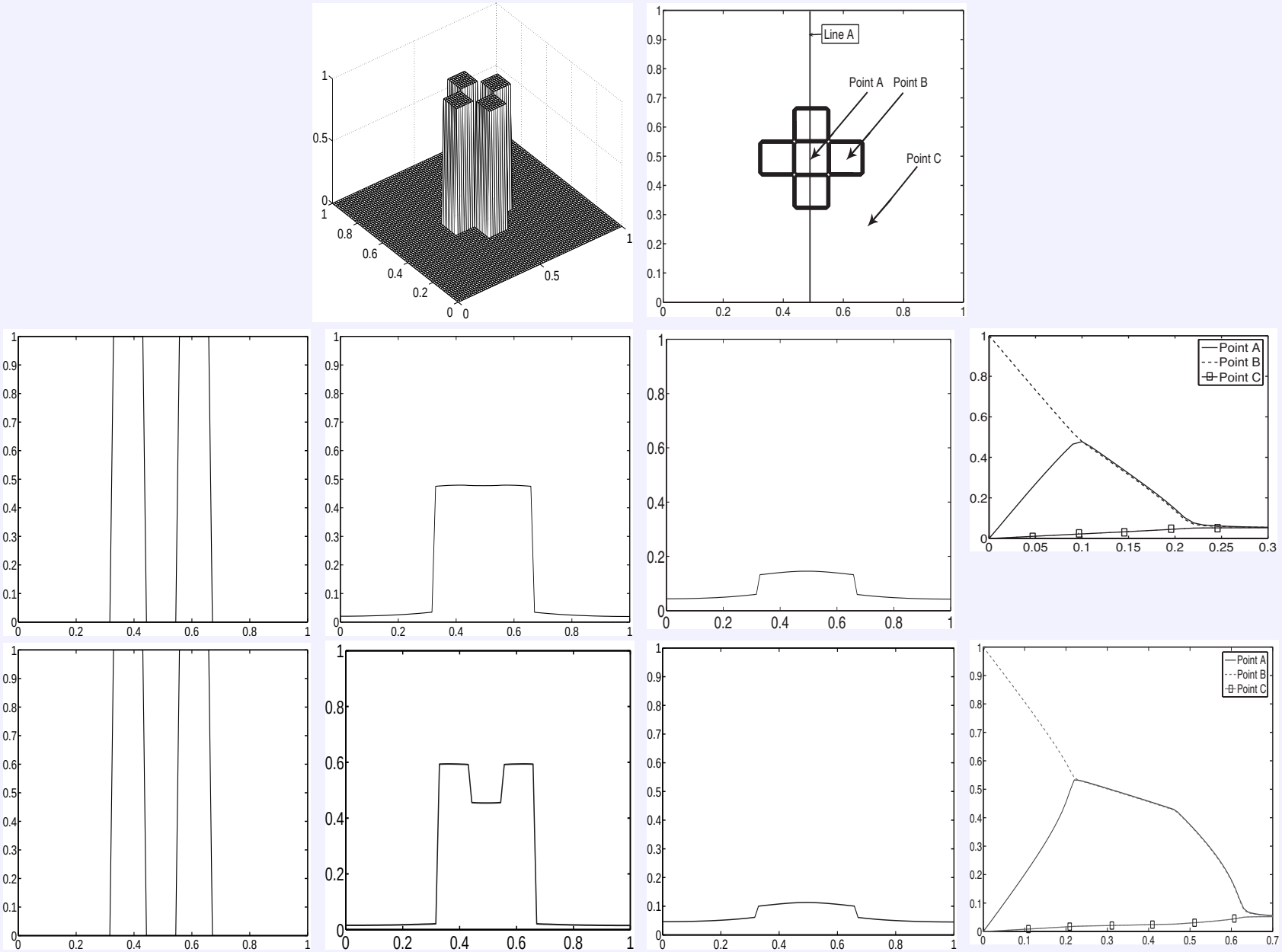
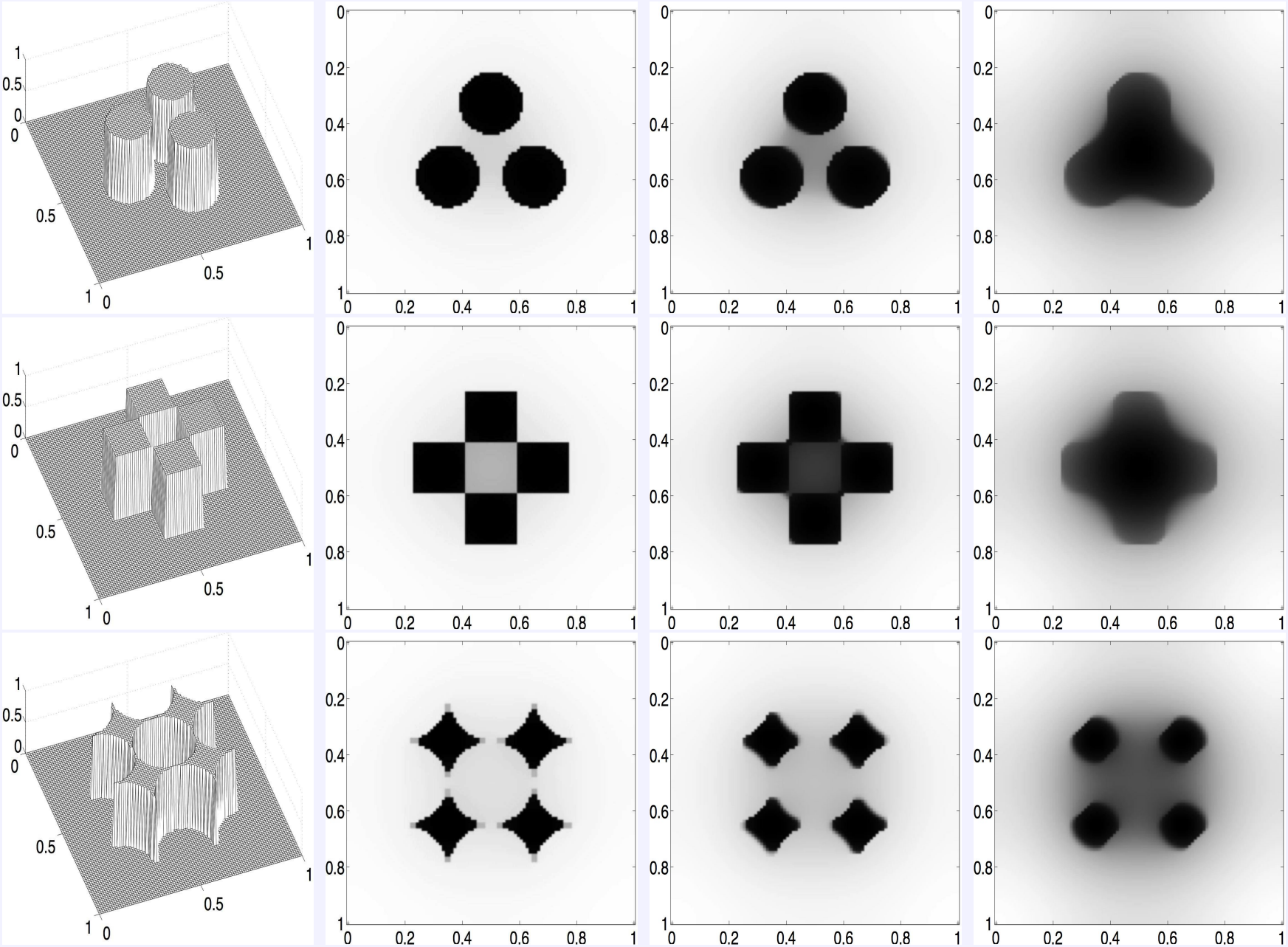
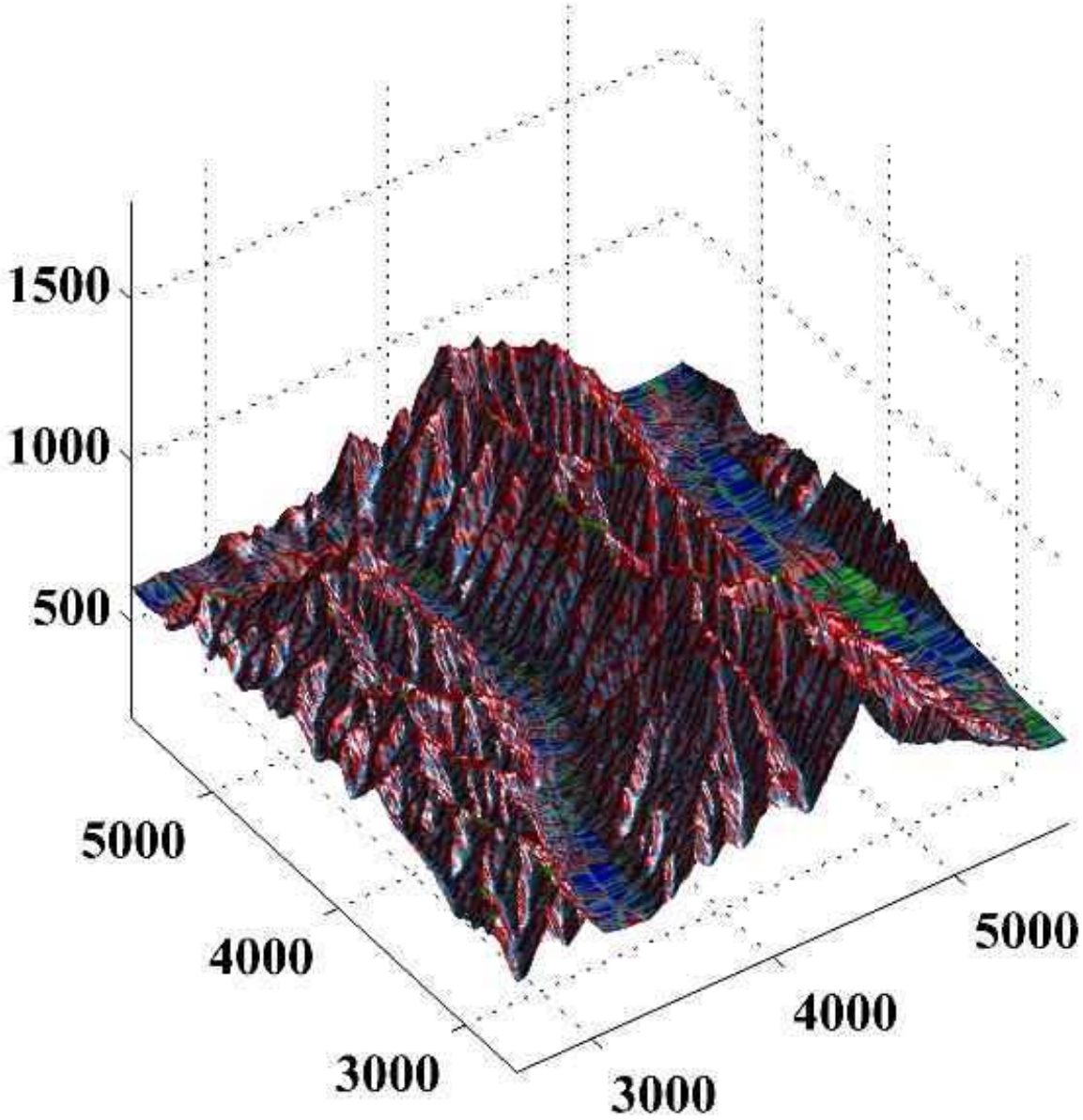


Perona-Malik case TV-Flow case

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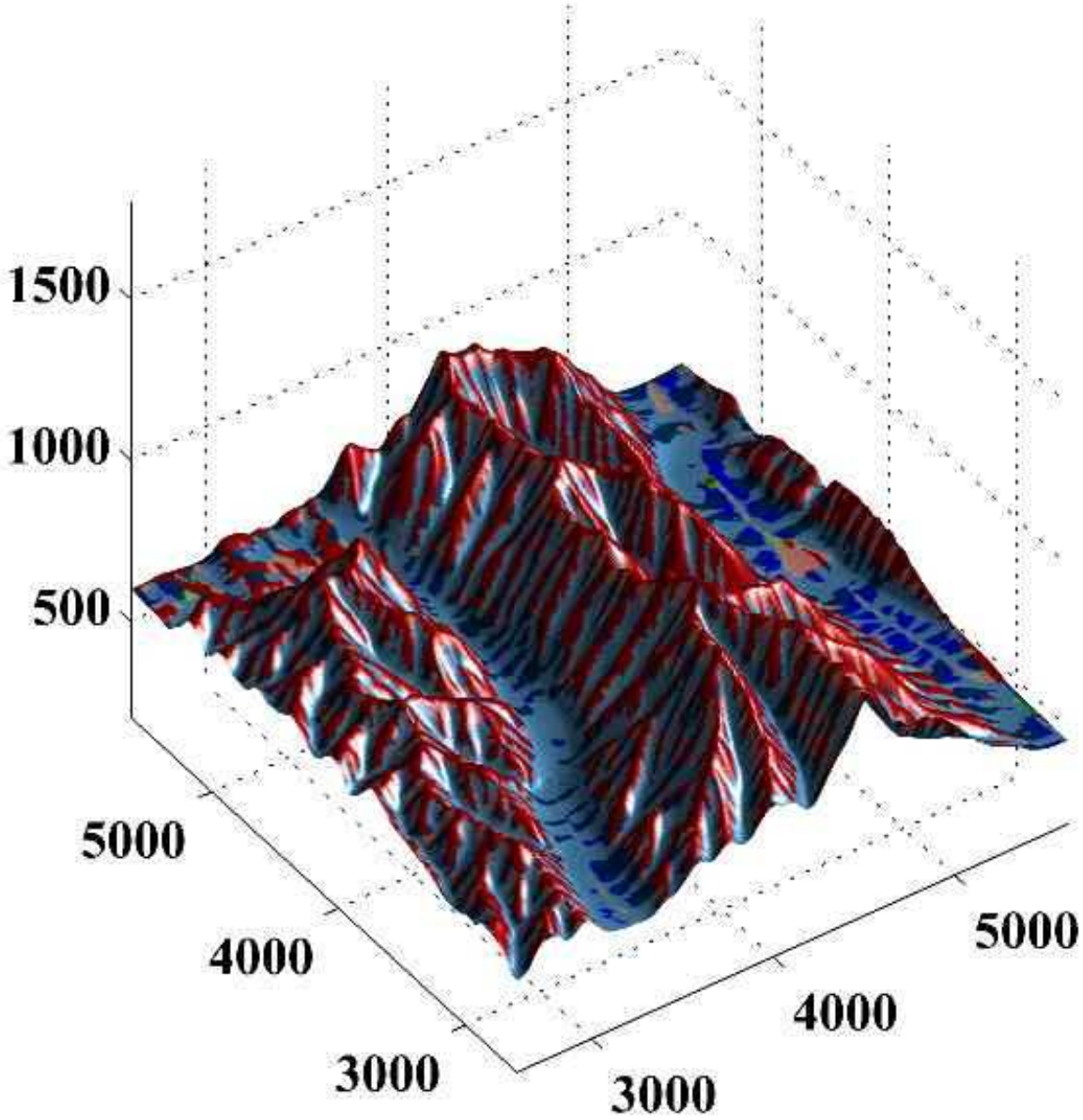
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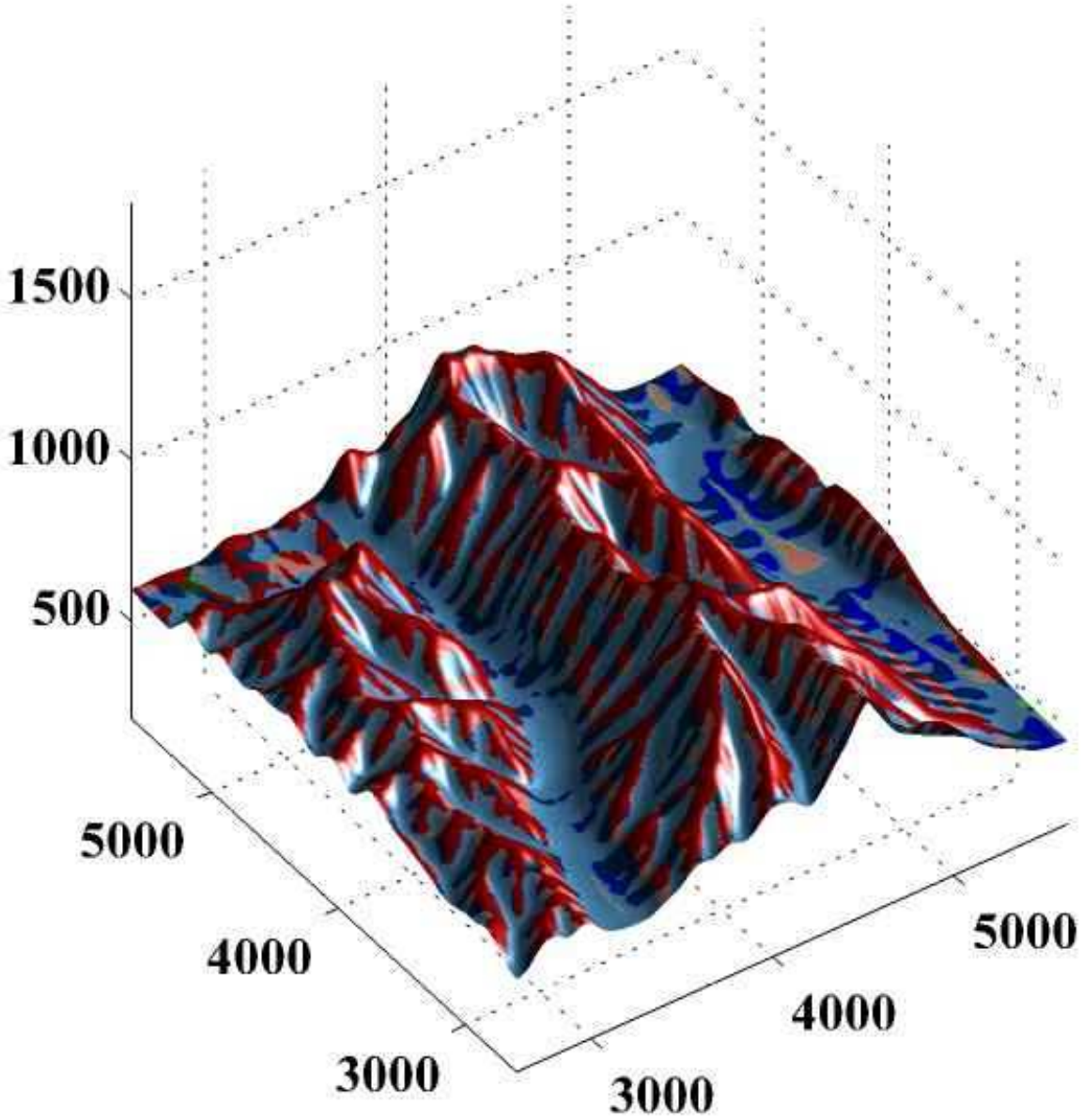
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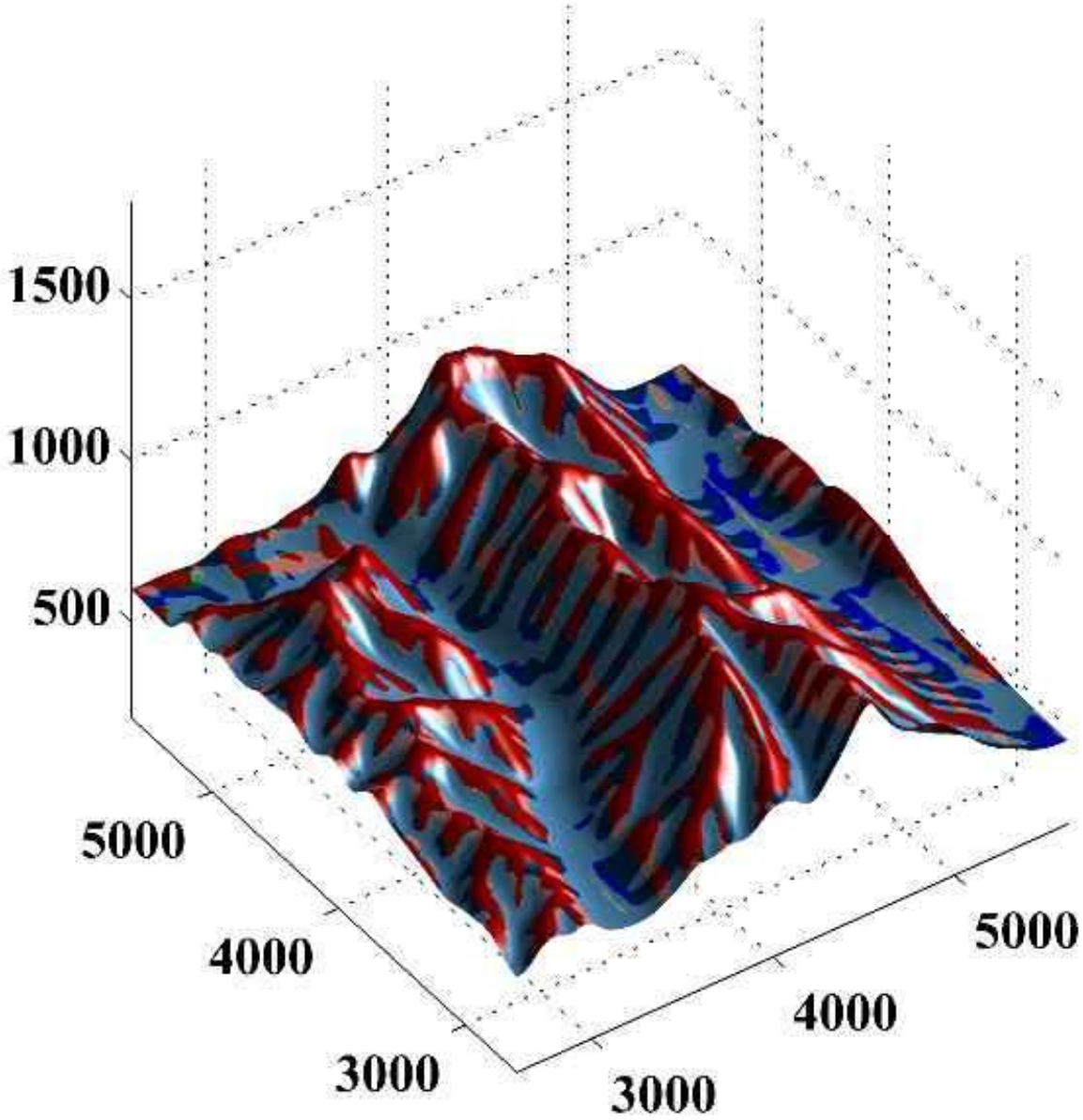
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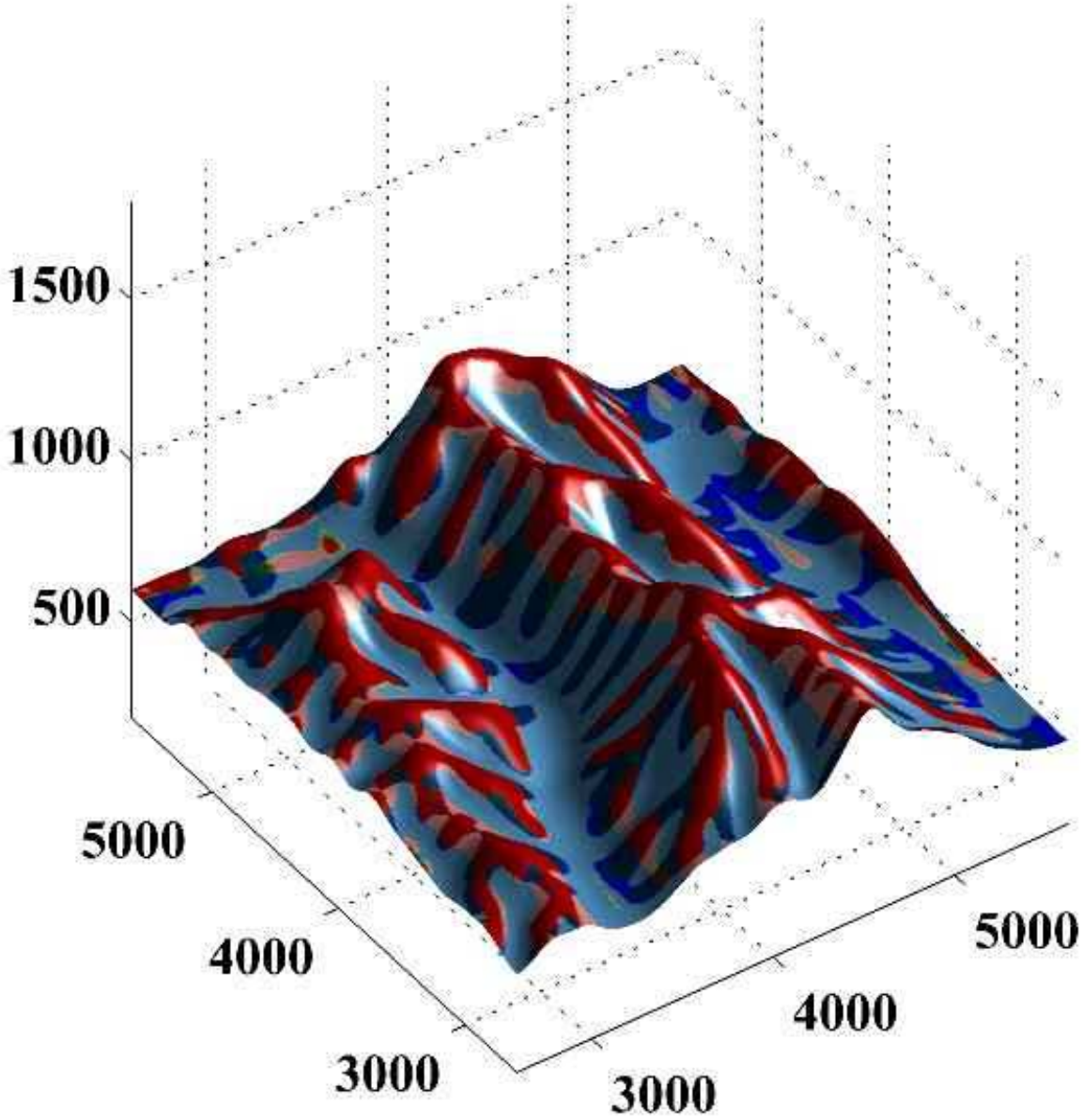
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Summary

- Motivation from image processing
- Forward-backward equations are not well posed
- Minimization corresponds to a non-convex optimization problem
- Young-measure is one mathematical tool to deal with such problems